Yang-Lee Zeros of the Two- and Three-State Potts Model Defined on ϕ^3 Feynman Diagrams

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February 1, 2008

Abstract

We present both analytic and numerical results on the position of the partition function zeros on the complex magnetic field plane of the $q = 2$ (Ising) and $q = 3$ states Potts model defined on ϕ^3 Feynman diagrams (thin random graphs). Our analytic results are based on the ideas of destructive interference of coexisting phases and low temperature expansions. For the case of the Ising model an argument based on a symmetry of the saddle point equations leads us to a nonperturbative proof that the Yang-Lee zeros are located on the unit circle, although no circle theorem is known in this case of random graphs. For the $q = 3$ states Potts model our perturbative results indicate that the Yang-Lee zeros lie outside the unit circle. Both analytic results are confirmed by finite lattice numerical calculations.

PACS-No.: 05.50.+q; 05.70.Fh; 64.60.Cn; 75.10.Hk

Keywords: Phase transitions; Critical phenomena; partition function zeros; Yang-Lee theory; Ising model; Potts model; Random matrix

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1 Introduction

The Ising model is known to be exactly solvable in one dimension and also in two dimensions in the absence of magnetic field. The magnetic field breaks the $\mathcal{Z}_2$ symmetry which complicates a possible exact solution of the model. However, it is still possible to prove for arbitrary temperature an *exact* theorem (see [1]) about the location of the zeros of the partition function on the complex fugacity $u = e^{2H}$ plane, where H stands for a complex magnetic field measured in units of $1/\beta = T$. Henceforth, these zeros will be called Yang-Lee zeros. The theorem establishes that the Yang-Lee zeros are located on the unit circle $|u_k| = 1, \quad k = 1, \cdots n$. Here n is the number of sites of the lattice whose details like coordination number and topology are not important for the proof. The symmetry under a change of sign of the magnetic field $u \to 1/u$ is a key ingredient but it is not sufficient for the proof. The proof is a bit technical and strongly depends on the form of the polynomials $P_n(u)$ in the partition function. A simpler proof would certainly be welcome. The authors of [2] have suggested an analytic way to find the location of the partition function zeros. It is assumed that for a system defined in a periodic volume $V = L^d$ with r different phases it is possible to write down the partition function of the system in terms of some functions f_l $(l = 1, \cdots, r)$ as follows:

$$\mathcal{Z} = \sum_{l=1}^{r} q_l e^{-\beta V f_l} + \mathcal{O}(e^{-L/L_0} e^{-\beta V f}). \tag{1}$$

where q_l is the degeneracy of the corresponding phase l and f_l is interpreted as its metastable free energy. The quantity L_0 is of the order of the correlation length while $f = \min \Re e(f_k)$. Following [2], we can use a destructive interference condition in order to find the zeros of $\mathcal{Z}$ within $\mathcal{O}(e^{-L/L_0})$:

$$\Re e\, f_{l,L}^{\text{eff}} = \Re e\, f_{m,L}^{\text{eff}} < \Re e\, f_{k,L}^{\text{eff}} \quad \text{for all} \quad k \neq l, m \tag{2}$$

$$\beta V \left(\Im m\, f_{l,L}^{\text{eff}} - \Im m\, f_{m,L}^{\text{eff}} \right) = \pi \mod 2\pi \tag{3}$$

where $f_l^{\text{eff}} = f_l - (\beta V)^{-1} \log q_l$. It is assumed that the metastable free energies are functions of some complex parameter z which is in our case the magnetic field. The continuity of the real part of the free energy across the line of zeros already appeared in [3]. It is clear that equations (2) and (3) are useful whenever we have a closed expression for the free energies of the different phases of the system. In the case of the two dimensional Ising model in the presence of a magnetic field, although we do not have exact expressions for the free energies we can use low temperature expansions (LTE's) and the symmetry $H \to -H$ as in [2] to furnish a simple proof of the Lee-Yang theorem valid in the thermodynamic limit and for low temperatures. In this work we apply the same ideas for a particular case of fluctuating random lattice. For such lattices no circle theorem has been proven. The spins degrees of freedom are placed on vertices of Feynman diagrams which

are themselves dynamical degrees of freedom and need to be summed over in the partition function with appropriate combinatorial factors. In the case where the diagrams have double lines and are generated by a random $N \times N$ matrix model (see [4, 5]) their sum mimics, in the continuum limit, the Ising model on random surfaces (two dimensional gravity). Early [6, 7] and recent [8] numerical results on graphs of planar topology indicate that the Yang-Lee zeros lie on the unit circle although no circle theorem exists in this case. Notice that the form of the polynomials used in the proof of the Lee-Yang theorem is not preserved by their linear combinations. However, we believe that there might a generalization of the circle theorem for the new class of polynomials obtained by linear combinations with positive coefficients of partition functions of the Ising type. In fact, further support to this conjecture has been given in [9] where we studied the effect of the topology of the graphs on the position of the Yang-Lee zeros. We have noticed that changing from the planar to the torus topology the Yang-Lee zeros remained on the unit circle with a tiny change in their positions which is in agreement with the existence of an underlying robust theorem. Furthermore, in order to take the continuum limit we take large matrices $N \to \infty$ which allows one to find (see [4]) with help of the orthogonal polynomial technique a closed expression for the free energy even in the presence of the magnetic field. Thus, an analytic study of the free energy singularities is possible and leads to pure imaginary values for the magnetic field (see [6]), i.e., $|u_k| = 1$. Of course, this is only correct in the thermodynamic limit. On the other hand, the same issue of Yang-Lee zeros on a dynamical lattice can be studied in a simplified model of thin graphs where matrices become numbers ($N = 1$) and our final partition function becomes a linear combination of partition functions on single line Feynman diagrams. In this case no random surface interpretation exists and the model exhibits a ferromagnetic phase transition with mean field exponents, see [10] and references therein. Once again numerical results for finite number of vertices n indicate the unit circle as the locus of Yang-Lee zeros [9]. However, since we do not have a random surface interpretation, the results of [6] do not go through the case of thin graphs. In particular, the location of the Yang-Lee zeros is not known even in the thermodynamic limit. In this work we analyze this question starting from a closed expression for the free energy of the model obtained in the thermodynamic limit by means of a saddle point approximation both for the Ising and the Potts model. Then, we use the destructive interference equations of [2] to find out the location of the Yang-Lee zeros. Unfortunately the presence of the magnetic field complicates the form of the resulting expression for the free energy but an argument based on low temperature expansions and a symmetry ($H \to -H$) between different saddle point solutions will allow us to prove that the Yang-Lee zeros of the Ising model on thin graphs should be on the unit circle in the thermodynamic limit. The proof is *exact* for the whole range of temperatures for which the LTE's converge. For the case of the Potts model we have not been able to promote our low temperature results to an *exact* level but perturbatively we show that the Yang-Lee zeros should lie outside (but close to) the unit circle for low temperatures. The precise location

of the zeros seem to change with the temperature in a complicated way. We also display numerical results in agreement with our analytic findings.

2 The Partition Function on Thin Graphs

We can define the q-state Potts model as a generalization of the Ising model ($q = 2$) where on each site of the lattice there is a spin degree of freedom σ which can take q different values $\sigma = 1, 2, \cdots, q$. By summing over all spin configurations $\{\sigma_i\}$ on a given static lattice G_n of n vertices we obtain the partition function $\mathcal{Z}_q(G_n)$. In the presence of a static magnetic field H in the $\sigma = 1$ direction we have,

$$\mathcal{Z}_q(G_n) = \sum_{\{\sigma_i\}} e^{\beta \sum_{<i,j>} \delta_{\sigma_i,\sigma_j} + 2H \sum_{i=1}^{n} \delta_{1,\sigma_i}} , \tag{4}$$

where $\sum_{<i,j>}$ is a sum over all bonds (propagators) of G_n including loop bonds which connect a site (vertex) to itself. Here we are interested in the case where the lattice is a dynamical degree of freedom and the final partition function is obtained by summing over all G_n with n vertices:

$$\mathcal{Z}_q^{(n)} = \sum_{k=1}^{K(n)} \mathcal{Z}_q(G_n^{(k)}) \tag{5}$$

In our case $K(n)$ will be the total number of Feynman graphs $G_n^{(k)}$ with n cubic vertices ϕ^3 and no external legs. This clearly includes graphs of different topologies. Each $\mathcal{Z}_q(G_n^{(k)})$ as well as $\mathcal{Z}_q^{(n)}$ will be proportional to a polynomium $P_n(u)$ in the fugacity $u = e^{2H}$. We take cubic vertices since those are the simplest ones after the trivial quadratic case ϕ^2 which corresponds to a $D = 1$ lattice. The partition functions $\mathcal{Z}_q^{(n)}$ can be generated by integrating over q zero dimensional fields $X_1, \cdots, X_q$ representing the q different spin states $\sigma = 1, 2, \cdots, q$ respectively,

$$\mathcal{Z}_q^{(2n)} = \frac{T_n}{2\pi\imath} \oint \frac{dg}{g^{2n+1}} \left(\frac{\int \mathcal{D}\mu \, e^{-\frac{1}{2}\left[\sum_{i=1}^{q} X_i^2 - 2c\sum_{i>j} X_i X_j - \frac{2g}{3}(e^{2H} X_1^3 + \sum_{i=2}^{q} X_i^3)\right]}}{\int \mathcal{D}\mu \, e^{-\frac{1}{2}\left[\sum_{i=1}^{q} X_i^2 - 2c\sum_{i>j} X_i X_j\right]}} \right) \tag{6}$$

where $\mathcal{D}\mu = \prod_{i=1}^{q} dX_i$ and T_n is a numerical factor independent of the magnetic field and temperature which is necessary to get rid of the bad asymptotic behavior of the undecorated ϕ^3 graphs. The contour integral is necessary to single out the term g^{2n} containing $2n$ cubic vertices. Notice that $\mathcal{Z}_q^{(n)}$ vanishes for odd number of vertices, therefore we write the total number of vertices as $2n$. The constant c will be related to the temperature. To each cubic vertex (X_1^3) representing a spin in the direction of the magnetic field there will be a $u = e^{2H}$ factor in agreement with the paramagnetic interaction term in (4). The quadratic terms in the argument of the exponentials in eq. (6) are responsible for the free propagators $< X_i X_j >$ (or bonds $< \sigma_i \sigma_j >$)

between the q different vertices associates with the states σ_i. Taking ratios of the propagators and comparing with the ratios of the corresponding Boltzmann weights of (4) we identify the parameter c with a function of the temperature as follows. The action in the numerator of eq. (6) can be written as

$$S_g \,=\, \frac{1}{2}\sum_{i,j=1}^{q} X_i K_{ij} X_j - \frac{g}{3}\,(e^{2H}\, X_1^3 + \sum_{i=2}^{q} X_i^3)\,, \tag{7}$$

where we have introduced the kinetic $q \times q$ matrix $K_{ij} = -c, \text{if } i \neq j$ and $K_{ii} = 1$. We can check that $\det K = (1+c)^{q-1}(1-(q-1)c)$. The propagators are given by the elements of the inverse matrix $K^{-1}_{ij} = (\det K)^{-1} c$ if $i \neq j$ and $K^{-1}_{ii} = (\det K)^{-1}\,[1-(q-2)c]$. Therefore, using the Boltzmann weights from (4) at vanishing magnetic field we have

$$\frac{\langle X_i X_j \rangle}{\langle X_i X_i \rangle} \,=\, \frac{c}{[1-(q-2)c]} \,=\, \frac{e^{E_{ij}(H=0)}}{e^{E_{ii}(H=0)}} \,=\, e^{-\beta}, \tag{8}$$

and consequently

$$c \,=\, (e^{\beta} + q - 2)^{-1}\,. \tag{9}$$

Therefore corresponding to $0 \leq T \leq \infty$ we have respectively the compact range $0 \leq c \leq 1/(q-1)$.

In the thermodynamic limit $n \to \infty$ one can evaluate the free energy

$$f_q^{(2n)} \,=\, -\frac{1}{2n}\,\log \mathcal{Z}_q^{(2n)} \tag{10}$$

by means of a saddle point approximation as follows. The fist step is to rescale the zero dimensional fields $X_i \to \frac{1}{g}\tilde{X}_i$ such that $S_g(X_i,u,c) \to (1/g^2)\,S_{g=1}(\tilde{X}_i,u,c)$. Next, we decouple the contour integral over the coupling g from the integral over the zero dimensional fields by a change of variables $g \to \tilde{g}\sqrt{S_{g=1}}$. Finally, we have

$$f_q \,=\, \lim_{n\to\infty}\Big[-\frac{1}{2n}\,\log \int D\mu e^{-n\log S_{g=1}} \,+\, \mathcal{O}(\frac{1}{n})\Big] \tag{11}$$

Therefore the free energy per site is given in the thermodynamic limit by [11]:

$$f_q \,=\, \frac{1}{2}\,\log \tilde{S}_{g=1} \tag{12}$$

where $\tilde{S}_{g=1}$ corresponds to $S_{g=1}$ at a solution of the saddle point equations $\partial_{X_i} S_{g=1} = 0$. Henceforth it is always assumed $g = 1$ and we will treat the $q = 2$ (Ising) and $q = 3$ states Potts model separately.

3 Ising Model

Aligning the magnetic field in the direction of σ_1, which is represented by the field X_1, and using the notation $X_1 = x$ and $X_2 = y$ we have respectively the saddle point equations $\partial_x S = 0 = \partial_y S$:

$$
\begin{aligned}
x - cy &= ux^2 \\
y - cx &= y^2
\end{aligned}
\tag{13}
$$

Using those equations we can write down the action at the saddle point in a quadratic form:

$$\tilde{S} = \frac{1}{6}\left[x^2 + y^2 - 2cxy\right] \tag{14}$$

From (13) we have

$$x = (1/c)(y - y^2) \tag{15}$$

$$uy(1-y)^2 + c(y-1) + c^3 = 0 \tag{16}$$

In the absence of magnetic field ($u = 1$) the solutions of (16) simplify (see [13]) and we have two low temperature solutions below the critical value $c_{cr} = 1/3$ and a high temperature solution valid for the entire range $0 \le c \le 1$:

$$y_{LT\pm} = \frac{1}{2}\left[1 + c \mp \sqrt{(1-3c)(1+c)}\right] \tag{17}$$

$$y_{HT} = 1 - c \tag{18}$$

Defining the magnetization

$$m = \left\langle \frac{ux^3}{(ux^3 + y^3)} \right\rangle \tag{19}$$

we can check that y_{LT+} and y_{LT-} correspond respectively to $m > 1/2$ and $m < 1/2$. Both solutions collapse at $m(x_{HT}, y_{HT}) = 1/2$ for $c = 1/3$. Substituting (17), (18) and (15) back in (14) we have:

$$\tilde{S}_{LT} = \frac{(1+c)^2(1-2c)}{6} \tag{20}$$

$$\tilde{S}_{HT} = \frac{(1-c)^3}{3} \quad . \tag{21}$$

Notice that both low temperature solutions have furnished the same action $\tilde{S}_{LT}$.

The presence of a nonvanishing magnetic field ($u \neq 1$) makes the solution of the cubic equation (16) awkward and not very illuminating. We have solved (16) instead as a Taylor expansion around $c = 0$ (LTE). We found three expansions $y_{LT\pm}$ and y_{HT} which reduce to (17) and (18) at $u = 1$. They furnish the following actions :

$$\tilde{S}_{LT+} = \frac{1}{6u^2} - \frac{c^2}{2u^2} - \frac{c^3}{3u^3} + \frac{(u^2-1)c^4}{2u^4} + \mathcal{O}(c^5) \tag{22}$$

$$\tilde{S}_{LT-} = \frac{1}{6} - \frac{c^2}{2} - \frac{uc^3}{3} - \frac{(u^2-1)c^4}{2} + \mathcal{O}(c^5) \tag{23}$$

$$\begin{aligned} \tilde{S}_{HT} &= \frac{1+u^2}{6u^2} - \frac{c}{u} + \frac{(1+u^2)c^2}{2u^2} + \frac{(u^4-3u^2+1)c^3}{3u^3} \\ &+ \frac{(u^2-1)^2(1+u^2)c^4}{2u^4} + \mathcal{O}(c^5) \end{aligned} \tag{24}$$

Clearly $\tilde{S}_{LT+}$ and $\tilde{S}_{LT-}$ collapse into (20) at $u=1$. For $u>1\,(H>0)$ $f_{LT+} < f_{LT-}$ and the system is magnetized in the direction of the magnetic field (x-direction) while the situation is reversed for $u<1\,(H<0)$ as expected. In order to locate the Yang-Lee zeros of $\mathcal{Z}^{(2n)}_{q=2}$ in the thermodynamic limit we first assume $u=\rho e^{\imath\theta}$ and then solve equations (2) and (3):

$$\Re e\,(f_{LT+} - f_{LT-}) = 0 \tag{25}$$

$$\Im m\, f_{LT-} = \Im m\, f_{LT+} + \frac{(2k-1)}{\beta V \pi} \tag{26}$$

There are two basic ingredients which will allow us to locate *exactly* the Yang-Lee zeros for the whole convergence range of the LTE's without really worrying about the numerical details of the coefficients of the expansions (22-24). First of all, notice that for $H\to-\infty$ (or $u\to 0$) the cubic equation (16) has a unique solution $y(u\to 0) = 1-c^2$ and therefore there must be at least one LTE which is well behaved for $u\to 0$ and terminate at the term proportional to c^2. Of course, this can only be identified with $\tilde{S}_{LT-}$ which implies that their coefficients can only involve positive powers of u, i. e. ,

$$\tilde{S}_{LT-}(u,c) = \sum_{n=0}^{\infty} a_n(u)c^n \quad . \tag{27}$$

Where each $a_n(u)$ is a polynomial in u. The next important ingredient is a symmetry of the saddle point equations (13) which relate solutions with $H>0$ to solutions with $H<0$. Namely, (13) is invariant under

$$u \to 1/u \tag{28}$$

$$x_i(1/u,c) \to u y_j(u,c) \tag{29}$$

$$y_i(1/u,c) \to u x_j(u,c) \tag{30}$$

where (x_i,y_i) and (x_j,y_j) are, in principle, distinct solutions of (13). Back in (14) we are led to the relation:

$$\tilde{S}_i\left(\frac{1}{u},c\right) = u^2\,\tilde{S}_j(u,c) \tag{31}$$

The labels i, j indicate that the saddle point solutions on both sides of (31) do not need to be the same. From the first terms of the LTE's in (22)-(24) and (31) we have the identification below

$$\tilde{S}_{LT+}(u,c) = \frac{1}{u^2}\,\tilde{S}_{LT-}\left(\frac{1}{u},c\right) \tag{32}$$

$$\tilde{S}_{HT}(u,c) = \frac{1}{u^2}\,\tilde{S}_{HT}\left(\frac{1}{u},c\right) \tag{33}$$

Putting back in the definition of the free energy and using (27) we obtain

$$\begin{aligned} 2(f_{LT-} - f_{LT+}) &= \log u^2 \\ &+ \left(\log \sum_{n=0}^{\infty} a_n(u)c^n - \log \sum_{n=0}^{\infty} a_n\Big(\frac{1}{u}\Big)c^n\right) \end{aligned} \tag{34}$$

Using $a_0 = 1/6$ and $u = \rho e^{\imath\theta}$, after expanding the logarithms we deduce:

$$2\Re e\,(f_{LT-} - f_{LT+}) = 2\log\rho + \sum_{n=1}^{\infty} c^n \sum_{k=1}^{B(n)} A_{k,n}\cos(k\theta)\left(\rho^k - \frac{1}{\rho^k}\right) \tag{35}$$

where $B(n)$ and $A_{k,n}$ are pure numerical factors. Thus, the condition $\Re e\,(f_{LT-} - f_{LT+}) = 0$ imply that the zeros must be on the unit circle $u_k = e^{\imath\theta_k}$ in the thermodynamic limit. The condition on the imaginary parts of the free energies will locate the corresponding angles θ_k on the circle as a function of the temperature. We emphasize that our proof of the circle theorem for thin ϕ^3 graphs hold inside the convergence region of low temperature expansions.

4 $q = 3$ States Potts Model

Using the notation $X_1 = x, X_2 = y, X_3 = z$ and fixing the magnetic field once again along the x-direction we derive the saddle point equations, $\partial_{X_i} S = 0$:

$$x - c(y+z) = ux^2 \tag{36}$$

$$y - c(x+z) = y^2 \tag{37}$$

$$z - c(x+y) = z^2 \tag{38}$$

From the difference of (37) and (38) we deduce that there are two groups of solutions, either $y = z$ or $y + z = 1 + c$:

I)

$$z = y \tag{39}$$

$$x = \frac{y}{c}(1 - c - y) \tag{40}$$

$$uy(y + c - 1)^2 + c(y + c - 1) + 2c^3 = 0 \tag{41}$$

II)

$$z = 1 + c - y \tag{42}$$

$$x_{\pm} = \frac{1 \pm [1 - 4uc(1 + c)]^{1/2}}{2u} \tag{43}$$

$$y^2 - (1 + c)y + c(1 + c + x) = 0 \tag{44}$$

As in the Ising case it is instructive to first look at the solutions for vanishing magnetic field ($u = 1$). In this case we have from the first set of the solutions two low temperature and one high temperature solution :

$$\begin{aligned} y_{LT\pm} &= z_{LT\pm} = \frac{1 \mp [1 - 4c(1 + c)]^{1/2}}{2} \\ x_{LT\pm} &= \frac{1 + 2c \pm [1 - 4c(1 + c)]^{1/2}}{2} \end{aligned} \tag{45}$$

$$x_{HT} = y_{HT} = z_{HT} = 1 - 2c \tag{46}$$

Defining the magnetization

$$m(q = 3) = \left\langle \frac{ux^3}{ux^3 + y^3 + z^3} \right\rangle \tag{47}$$

We identify the labels $LT+$ and $LT-$ with solutions such that $m > 1/3$ and $m \leq 1/3$ respectively. The solution $LT-$ meets the high temperature solution with $m = 1/3$ at $c = 1/5$. The second group of solutions (43) does not furnish for $u = 1$ any new physical solution. Explicitly, for $0 \leq c \leq (\sqrt{2} - 1)/2$, we have from the second group two real solutions:

$$x_1 = y_1 = \frac{1 - [1 - 4c(1 + c)]^{1/2}}{2} \tag{48}$$

$$z_1 = \frac{1 + [1 - 4c(1+c)]^{1/2}}{2} \tag{49}$$

and

$$x_2 = \frac{1 + [1 - 4c(1+c)]^{1/2}}{2} \tag{50}$$

$$y_2 = \frac{1 + c + |[1 - 4c(1+c)]^{1/2} - c|}{2} \tag{51}$$

$$z_2 = \frac{1 + c - |[1 - 4c(1+c)]^{1/2} - c|}{2} \tag{52}$$

Notice that solution (x_1, y_1, z_1) corresponds to a permutation $(x, z) \to (z, x)$ of the solution $LT+$ given in (45) while the solution (x_2, y_2, z_2) coincides for $0 \le c \le 1/5$ with the permutation $(x, z) \to (z, x)$ of the solution $LT-$. However, its derivative is discontinuous at $c = 1/5$ and the solution deviates from $LT-$ for $1/5 < c < (\sqrt{2} - 1)/2$. Concluding, we can safely neglect both solutions of the second group.

Turning on the magnetic field ($u \neq 1$), the solutions of the cubic equation in (40) become cumbersome. Once again we make a low temperature expansion around $c = 0$ which we display below after the substitution in the action. From the first group of solutions ($y = z$) we have three possibilities:

$$\begin{aligned} \tilde{S}_0 &= \frac{1+2u^2}{6u^2} - c\frac{2+u}{u^2} + \frac{c^2}{u^3}(1 + 2u + 3u^2) \\ &+ \frac{c^3}{3u^4}(8u^4 - 13u^3 - 12u^2 + 2u + 2) + \mathcal{O}(c^4). \end{aligned} \tag{53}$$

$$\tilde{S}_1 = \frac{1}{6u^2} - \frac{c^2}{u^2} - \frac{c^3(2+3u)}{3u^3} + \mathcal{O}(c^4) \tag{54}$$

$$\tilde{S}_2 = \frac{1}{3} - c - c^2 + \frac{c^3(11 - 8u)}{3} + \mathcal{O}(c^4) \tag{55}$$

Expanding the second group of solutions $(y + z = 1 + c)$ we have two possibilities :

$$\tilde{S}_3 = \frac{1}{6} - c^2 - \frac{c^3(u+4)}{3} + \mathcal{O}(c^4) \tag{56}$$

$$\tilde{S}_4 = \frac{1+u^2}{6u^2} - \frac{c}{u} - \frac{c^2}{u} - \frac{c^3(2+u)}{3} + \mathcal{O}(c^4) \tag{57}$$

Next we analyze the physical content of all those solutions. First, we notice that in the limit of vanishing magnetic field the only solution which correctly reproduces the LTE of $\tilde{S}_{HT}(u = 1, c)$ is $\tilde{S}_0$ and therefore we will identify it with the LTE of

$\tilde{S}_{HT}(u,c)$. However, the limit $u \to 1$ can not be used to identify the remaining solutions since the solutions of the second group $\tilde{S}_3$ and $\tilde{S}_4$ become identical respectively to $\tilde{S}_1$ and $\tilde{S}_2$ for $u \to 1$ which on their turn correspond to the LTE of the physical solutions $\tilde{S}_{LT+}$ and $\tilde{S}_{LT-}$ of the $H = 0$ case [1]. In order to understand the meaning of $\tilde{S}_1 - \tilde{S}_4$ we have to look at the limits $H \to +\infty$ ($u \to \infty$) and $H \to -\infty$ ($u \to 0$). The only LTE solution well behaved for $u \to \infty$ is $\tilde{S}_1$ which will be therefore identified with a strongly magnetized low temperature solution, i.e., for this solution we expect $\lim_{c\to 0} m = 1$. For $u \to 0$ both $\tilde{S}_2$ and $\tilde{S}_3$ are well behaved and good candidates but $\tilde{S}_3$ possess the lowest free energy (recall $f \sim \log S$) as one can check numerically. Therefore $\tilde{S}_3$ will represent the weakly magnetized phase which satisfies $\lim_{c\to 0} m = 0$. Thus, we have for the free energies of the physical low temperature solutions:

$$\begin{aligned}
2(f_{LT-} - f_{LT+}) &= \log \tilde{S}_3 - \log \tilde{S}_1 \\
&= \log u^2 + \left(\frac{4}{u} - 2u - 2\right) c^3 \\
&+ 3c^4 \left(\frac{2}{u^2} - u^2 + \frac{4}{u} - 2u - 3\right) + \mathcal{O}(c^5)
\end{aligned} \tag{58}$$

We can find the position of the Yang-Lee zeros plugging $u = \rho e^{\imath\theta}$ in (58) and solving the destructive interference equations (25) and (26). First, if we look at the leading terms (zero temperature) we will see from the equations (25) and (26) that the Yang-Lee zeros, in the thermodynamic limit, will be located on the unit circle $\rho = 1$ at $T = 0$. Our finite size numerical calculations are in agreement with those expectations (see Figure 1). Adding the next to leading terms of order c^2 we will still have the zeros on the unit circle but if we further truncate the LTEs at c^3 level the zeros will slightly move out the unit circle. Indeed, from (25) and (58) we have at order c^3 :

$$\cos\theta = \frac{\rho[c^3 - \log\rho]}{c^3(2-\rho^2)} \tag{59}$$

Imposing that $-1 \le \cos\theta \le 1$ we find numerically for $0 \le c \le .5$ that $1 \le \rho \le \rho^*$ where ρ^* increases with the temperature. It is worth commenting that the r.h.s. of (59) is a monotonically decreasing function of ρ. Consequently, we have a closed curve outside the unit circle which is nonsymmetric across the imaginary axis with the farthest point from the origin being $(\rho = \rho^*, \theta = \pi)$. In Figure 2 both our numerical and analytic results are overlapped. The finite size results seem to tend to the (full line) analytic one in the thermodynamic limit. In Figure 2 we have used the truncation at order c^4 instead of (59), although they differ by a tiny amount,

[1] Incidentally, we notice that the discontinuous behavior of the solution (22) only appears for $1/5 < c < (\sqrt{2}-1)/2$ which is outside the convergence region of the LTE of (22). Remember that for $0 \le c \le 1/5$ both $\tilde{S}_{LT-}$ and (22) are identical, which explains why $\tilde{S}_4$ and $\tilde{S}_2$ become equal at $u = 1$ and low temperatures.

including the c^4 terms make the analytic results closer to the numerical ones. The c^4 terms are quite complicated to be displayed here explicitly.

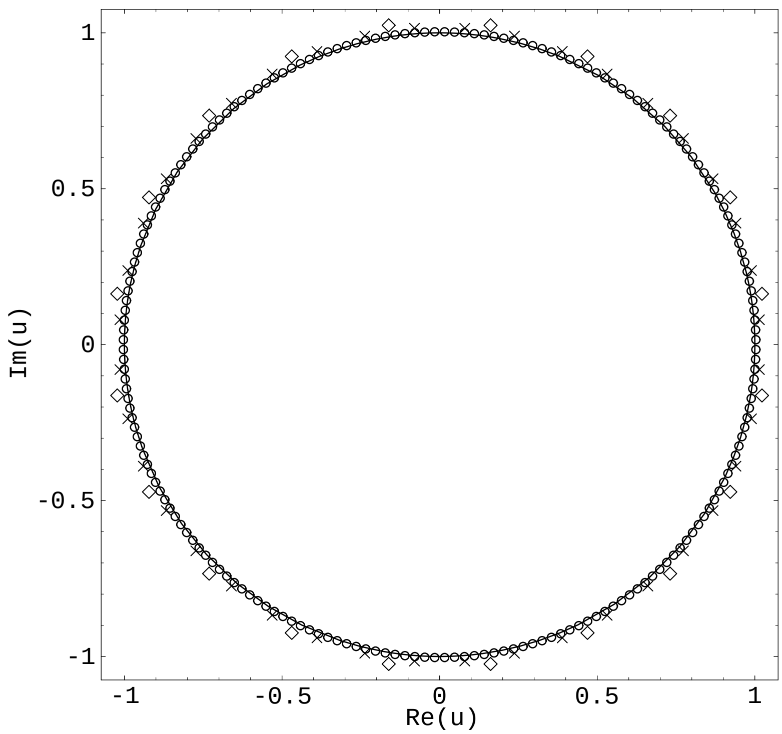

Fig. 1. Yang-Lee zeros of $\mathcal{Z}_{q=3}^{(2n)}$ at zero temperature, $c = 0$, for graphs with $2n = 20$ ($\diamond$), $2n = 40$ ($\times$) and $2n = 200$ ($\circ$) vertices. The closed curve is the unity circle.

5 $T \to \infty$

The results of previous two sections hold for the range of temperatures for which the LTE's converge. In general, we have not been able to go beyond those temperatures analytically. One exception is the case $T \to \infty$ (or $c \to 1/(q-1)$) for which a subtle decoupling of the zero dimensional fields take place for arbitrary q similarly to what happens for the Ising model ($q = 2$), see [9]. The key point is to take a particular scaling for the coupling g appearing in the action (7) as $c \to 1/(q-1)$. Namely, the matrix K_{ij} appearing in (7) has $q-1$ degenerate eigenvalues $\lambda_j = 1 + c\,(j = 1, 2, \cdots, q-1)$, and the nondegenerate one $\lambda_q = 1 - (q-1)c$. In order to decouple the fields X_j in the action it is natural to diagonalize the ma-

trix K_{ij} through an orthogonal transformation[2] $X_i \to A_{ij} X_j$ and rescale $X_i \to \frac{\tilde{X}_i}{\sqrt{\lambda_i}}$. The Jacobian cancel out in the calculation of $\mathcal{Z}_q^{(2n)}$. After those changes the action becomes (repeated indices are summed over)

$$S_g = \sum_{i=1}^{q} \frac{\tilde{X}_i^2}{2} - \frac{g}{3}\left[u \left(\frac{A_{1k}\tilde{X}_k}{\sqrt{\lambda_k}} \right)^3 + \sum_{j=2}^{q} \left(\frac{A_{jk}\tilde{X}_k}{\sqrt{\lambda_k}} \right)^3 \right] . \tag{60}$$

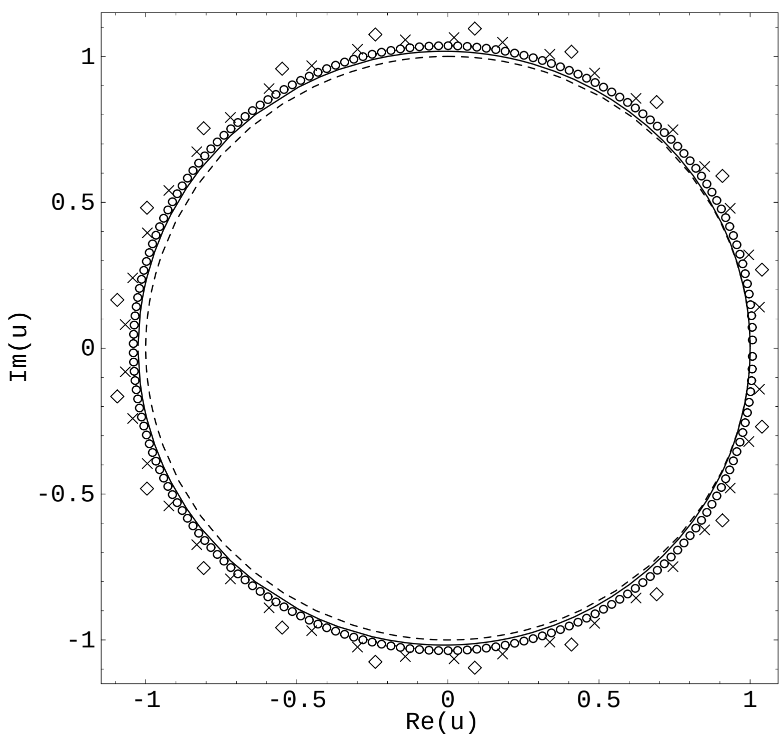

Fig. 2. Yang-Lee zeros of $\mathcal{Z}_{q=3}^{(2n)}$ at $c = 0.2$, for graphs with $2n = 20$ ($\diamond$), $2n = 40$ ($\times$) and $2n = 200$ ($\circ$) vertices, compared to the saddle point result (solid line) and to the unity circle (dashed line).

Apparently, we have just transferred the mixing terms to the higher powers of the potential, however in the limit $T \to \infty$ ($c \to 1/(q-1)$) we have $\lambda_q \to 0$, while all other eigenvalues remain finite. Thus, after the redefinition $g \to \overline{g}\lambda_q^{3/2}$, we have in that limit :

$$S_g(T \to \infty) = \sum_{i=1}^{q-1} \frac{\tilde{X}_i^2}{2} + \frac{\tilde{X}_q^2}{2} - \frac{\overline{g}}{3}\Big(u\, A_{1q}^3 + \sum_{j=2}^{q} A_{jq}^3 \Big) \tilde{X}_q^3 . \tag{61}$$

[2]Explicitly the orthogonal matrix reads $A_{ij} = 0$ if $i \geq j+2$, $A_{iq} = 1/\sqrt{q}$ and $A_{ij} = -1/\sqrt{j+j^2}$ if $i \leq j \leq q-1$ or $j/\sqrt{j+j^2}$ if $i = j+1 \leq q$.

Thus, we have decoupled the zero dimensional fields on a specific point in the parameter space while keeping information about nontrivial interacting terms. Clearly, this procedure is only useful if we do not care about the singular behavior of the original variable X_q. This is precisely the case of (6). Indeed, the Jacobian from X_i to $\tilde{X}_i$ is singular in the limit $T \to \infty$ but it cancels out because of the ratio in eq. (6). Since $A_{iq} = 1/\sqrt{q}$ we obtain

$$S(T \to \infty) \; = \; \sum_{i=1}^{q} \frac{X_i^2}{2} - \frac{\overline{g}}{3q^{3/2}} (y + q - 1) X_q^3 \, . \tag{62}$$

Substituting this action in (6), the integrations over $X_1, \cdots, X_{q-1}$ cancel out leaving us with an expression which is finite in the limit $c \to 1/(q-1)$. From the $\overline{g}^{2n}$ term we have :

$$\mathcal{Z}_q^{(2n)}(T \to \infty) \; = \; \tilde{T}_n (u + q - 1)^{2n} \tag{63}$$

where $\tilde{T}_n$ is a numerical factor independent of the magnetic field and temperature. Therefore we conclude that the Yang-Lee zeros of the q-state Potts model on thin graphs coalesce exactly at the point $u = 1 - q$ as $T \to \infty$. The same result is valid on a static lattice [15]. Again our numerical calculations confirm this analytic proof.

6 Conclusion

We have proved that in the thermodynamic limit the zeros of the partition function of the Ising model on thin graphs lie exactly on the unit circle in the complex fugacity plane. Our proof is *exact* in the range of temperatures for which the low temperature expansions converge. For the case of the $q = 3$ Potts model we do not have the $H \to -H$ symmetry anymore and our results were fairly perturbative in the temperature. In that case the zeros lie on closed curves which depend on the temperature. Those curves lie outside the unit circle and tend to the circle as $T \to 0$. Our numerical results for small number of vertices seem to be in agreement with the analytic ones derived by means of the destructive interference formulas of [2]. The numerical results were very similar to the static lattice case treated in [15] for both the Ising and $q = 3$ Potts model. We should mention that part of the motivation for this work came from a recent progress (see [10] and [8]) on using the destructive interference formulas of [2] to find analytically the exact curves formed by the Fisher zeros (complex temperatures) of the q-state Potts model on thin graphs. However, the results of [10] and [8] were derived for $H = 0$ and the presence of the magnetic field complicates the form of the solutions such that we had to appeal to low temperature expansions. As a further work one might look at Yang-Lee zeros for complex temperatures as in [16] as well as other types of vertices. Finally, we mention that, as in [9], we have also looked at connected partition functions obtained by first taking the logarithm of the ratio of integrals in (6) and then making the contour integral. The results for the position of the

Yang-Lee zeros were qualitatively similar to the ones reported here. We hope to return in the future to the open problem of proving the circle theorem for the Ising model on random lattices with finite number of vertices.

7 Acknowledgements

The authors would like to thank Nelson A. Alves for useful discussions and collaboration in early stages of this work and also Prof. D.J. Johnston for some useful e-mail exchange. L.C.A. would like to thank the Mathematical Physics Department of USP at São Paulo for their kind hospitality. This work was partially supported by FAPESP, grants 00/03277-3 (L.C.A.) and 00/12661-1 (D.D.), and CNPq (D.D.).

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Figure captions

- **Fig. 1.** Yang-Lee zeros of $\mathcal{Z}_{q=3}^{(2n)}$ at zero temperature, $c = 0$, for graphs with $2n = 20$ ($\diamond$), $2n = 40$ ($\times$) and $2n = 200$ ($\circ$) vertices. The closed curve is the unity circle.

- **Fig. 2.** Yang-Lee zeros of $\mathcal{Z}_{q=3}^{(2n)}$ at $c = 0.2$, for graphs with $2n = 20$ ($\diamond$), $2n = 40$ ($\times$) and $2n = 200$ ($\circ$) vertices, compared to the saddle point result (solid line) and to the unity circle (dashed line).